

# COMPARISON THEOREMS FOR ELLIPTIC EQUATIONS ON UNBOUNDED DOMAINS

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Comparison theorems of Sturm's type will be obtained for the linear elliptic partial differential equations

$$(1) \quad lu = \sum_{i,j=1}^n D_i(a_{ij}D_ju) + 2 \sum_{i=1}^n b_i D_i u + cu = 0,$$

$$(2) \quad Lv = \sum_{i,j=1}^n D_i(A_{ij}D_jv) + 2 \sum_{i=1}^n B_i D_i v + Cv = 0$$

on unbounded domains  $R$  in  $n$ -dimensional Euclidean space  $E^n$ . The boundary  $P$  of  $R$  is supposed to have a piecewise continuous unit normal vector at each point. Points in  $E^n$  are denoted by  $x = (x^1, x^2, \dots, x^n)$  and differentiation with respect to  $x^i$  is denoted by  $D_i$ ,  $i = 1, 2, \dots, n$ . The coefficients  $a_{ij}$ ,  $b_i$ ,  $c$ ,  $A_{ij}$ ,  $B_i$ , and  $C$  in (1) and (2) are assumed to be real and continuous on  $R \cup P$ , and the matrices  $(a_{ij})$  and  $(A_{ij})$  symmetric and positive definite in  $R$ . A "solution" of (1) (or (2)) is supposed to be continuous in  $R \cup P$  and have uniformly continuous first partial derivatives in  $R$ , and all derivatives involved in (1) (or (2)) are supposed to exist, be continuous, and satisfy the differential equation at every point in  $R$ .

Some recent results of Clark and the author [2], [7] apply to bounded domains  $R$  in  $E^n$ . In the self-adjoint case  $b_i = B_i = 0$ ,  $i = 1, 2, \dots, n$ , the *variation* of  $lu$  is defined as

$$(3) \quad V[u] = \int_R \left[ \sum_{i,j=1}^n (a_{ij} - A_{ij}) D_i u D_j u + (C - c) u^2 \right] dx.$$

The following result [2] is typical of those to be extended to unbounded domains.

**THEOREM A.** *Let  $R$  be a bounded domain in  $E^n$  whose boundary  $P$  has a piecewise continuous unit normal. Suppose  $b_i = B_i = 0$  in (1), (2),  $i = 1, 2, \dots, n$ . If there exists a nontrivial solution  $u$  of  $lu = 0$  in  $R$  such that  $u = 0$  on  $P$  and  $V[u] \geq 0$ , then every solution of  $Lv = 0$  vanishes at some point in  $\bar{R}$ .*

In [7] the author extended this to the general second-order linear elliptic equations (1) and (2). Theorem A generalizes a theorem of Hartman and Wintner [4] in which the condition  $V[u] \geq 0$  is replaced by the pointwise conditions  $C \geq c$  and  $(a_{ij} - A_{ij})$  is positive semidefinite in  $\bar{R}$ . In the case  $n = 1$ , Theorem A reduces to

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Leighton's generalization [5] of the classical Sturm-Picone theorem. In this case, because the solutions of second order ordinary linear differential equations have only simple zeros, it is easy to obtain the following modification of Theorem A: If there exists a nontrivial solution of  $lu=0$  in  $(\alpha, \beta)$  such that  $u(\alpha)=u(\beta)=0$  and  $V[u]>0$ , then every solution of  $Lv=0$  has a zero in  $(\alpha, \beta)$ . In the case  $n=2$ , Protter [6] obtained pointwise conditions on the coefficients in (1) and (2) to ensure the conclusion of Theorem A.

Our purpose here is to extend Theorem A to unbounded domains in  $E^n$ . Apparently no general results are known even in the case  $n=1$ . Our results will constitute an extension of the Sturm-Picone theorem in 4 directions: (i) to  $n$ -dimensions; (ii) to nonselfadjoint differential equations; (iii) to coefficients satisfying a general condition of the type  $V[u]\geq 0$ ; and (iv) to unbounded domains.

However, differential equations of order higher than 2 and general boundary conditions will not be considered here.

Let  $D_a$  denote the  $n$ -disk  $\{x \in E^n: |x-x_0|<a\}$  and let  $S_a$  denote the bounding  $(n-1)$ -sphere, where  $x_0$  is a fixed point in  $E^n$ . Define

$$R_a = R \cap D_a, \quad P_a = P \cap D_a, \quad C_a = R \cap S_a.$$

Clearly there exists a positive number  $a_0$  such that  $R_a$  is a bounded domain with boundary  $P_a \cup C_a$  for all  $a \geq a_0$ .

Let  $Q[z]$  be the quadratic form in  $n+1$  variables  $z_1, z_2, \dots, z_{n+1}$  defined by

$$(4) \quad Q[z] = \sum_{i,j=1}^n A_{ij}z_i z_j - 2z_{n+1} \sum_{i=1}^n B_i z_i + Gz_{n+1}^2,$$

where the continuous function  $G$  is to be determined so that this form is positive semidefinite. The matrix  $Q$  associated with  $Q[z]$  has the block form

$$Q = \begin{pmatrix} A & -B \\ -B^T & G \end{pmatrix}, \quad A = (A_{ij}),$$

where  $B^T$  is the  $n$ -vector  $(B_1, B_2, \dots, B_n)$ . Let  $B_i^*$  denote the cofactor of  $-B_i$  in  $Q$ . Since  $A$  is positive definite, a necessary and sufficient condition for  $Q$  to be positive semidefinite is  $\det Q \geq 0$ , or

$$(5) \quad G \det(A_{ij}) \geq - \sum_{i=1}^n B_i B_i^*.$$

The proof is a slight modification of the well-known proof for positive definite matrices [3].

Let  $M_a$  be the quadratic functional defined by

$$(6) \quad M_a[u] = \int_{R_a} F[u] dx,$$

where

$$(7) \quad F[u] = \sum_{i,j} A_{ij} D_i u D_j u - 2u \sum_i B_i D_i u + (G-C)u^2.$$

Define  $M[u] = \lim_{a \rightarrow \infty} M_a[u]$  (whenever the limit exists). The domain  $\mathfrak{D}_M$  of  $M$  is defined to be the set of all real-valued continuous functions  $u$  in  $R \cup P$  such that  $u$  has uniformly continuous first partial derivatives in  $R_a$  for all  $a \geq a_0$ ,  $M[u]$  exists, and  $u$  vanishes on  $P$ . Define

$$(8) \quad [u, v]_a = \int_{C_a} u \sum_{i,j} A_{ij} n_i D_j v \, ds$$

where  $(n_i)$  is the unit normal to  $C_a$ ;

$$(9) \quad [u, v] = \lim_{a \rightarrow \infty} [u, v]_a$$

whenever the limit on the right side exists.

**LEMMA 1.** *Suppose  $G$  satisfies (5) in  $R$ . If there exists  $u \in \mathfrak{D}_M$  not identically zero such that  $M[u] < 0$ , then every solution  $v$  of  $Lv = 0$  for which  $[u^2/v, v] \geq 0$  vanishes at some point of  $R \cup P$ .*

**Proof.** Suppose to the contrary that there exists a solution  $v \neq 0$  in  $R \cup P$ . For  $u \in \mathfrak{D}_M$  define

$$\begin{aligned} X^i &= v D_i(u/v), \\ Y^i &= v^{-1} \sum_j A_{ij} D_j v, \quad i = 1, 2, \dots, n. \end{aligned}$$

The following identity in  $R$  will now be established:

$$(10) \quad \sum_{i,j} A_{ij} X^i X^j - 2u \sum_i B_i X^i + Gu^2 + \sum_i D_i(u^2 Y^i) = F[u] + u^2 v^{-1} Lv.$$

The left member of (10) is equal to

$$\begin{aligned} & \frac{1}{v^2} \sum_{i,j} A_{ij} (v D_i u - u D_i v) (v D_j u - u D_j v) - \frac{2u}{v} \sum_i B_i (v D_i u - u D_i v) \\ & + Gu^2 + \frac{2u}{v} \sum_{i,j} A_{ij} D_i u D_j v + \frac{u^2}{v^2} \sum_{i,j} (v D_i (A_{ij} D_j v) - A_{ij} D_i v D_j v). \end{aligned}$$

Since  $(A_{ij})$  is symmetric, this reduces easily to the right member of (10). Since  $Lv = 0$  in  $R$ ,

$$(11) \quad \int_{R_a} F[u] \, dx = \int_{R_a} \left[ \sum_{i,j} A_{ij} X^i X^j - 2u \sum_i B_i X^i + Gu^2 \right] dx + \int_{R_a} \sum_i D_i(u^2 Y^i) \, dx.$$

The first integrand on the right side is a positive semidefinite form by the hypothesis (5). Since  $u = 0$  on  $P_a$ , it follows from Green's formula that

$$\begin{aligned} \int_{R_a} \sum_i D_i(u^2 Y^i) \, dx &= \int_{P_a \cup C_a} \sum_i u^2 n_i Y^i \, ds \\ &= \int_{C_a} \frac{u^2}{v} \sum_{i,j} A_{ij} n_i D_j v \, ds. \end{aligned}$$

Hence (8) and (11) yield

$$\int_{R_a} F[u] \, dx \geq [u^2/v, v]_a.$$

Since  $[u^2/v, v] \geq 0$  by hypothesis,

$$M[u] = \lim_{a \rightarrow \infty} \int_{R_a} F[u] \, dx \geq 0.$$

This contradiction establishes Lemma 1.

**LEMMA 2 (SELF-ADJOINT CASE).** *Suppose  $B_i = 0$  in (2) and (7),  $i = 1, 2, \dots, n$ . If there exists  $u \in \mathfrak{D}_M$  not identically zero such that  $M[u] \leq 0$ , then every solution  $v$  of  $Lv = 0$  for which  $[u^2/v, v] \geq 0$  vanishes at some point of  $R \cup P$ .*

**Proof.** In this case we can take  $G = 0$ , and the first integrand on the right side of (11) is a positive definite form. Hence

$$\int_{R_a} \sum_{i,j} A_{ij} X^i X^j \geq 0,$$

equality holding iff  $X^i$  is identically zero for each  $i = 1, 2, \dots, n$ ; i.e.,  $u$  is a constant multiple of  $v$ . The latter cannot occur since  $u = 0$  on  $P$  and  $v \neq 0$  on  $P$ , and therefore

$$\int_{R_a} F[u] \, dx > [u^2/v, v]_a.$$

It follows that  $M[u] > 0$ , contrary to the hypothesis  $M[u] \leq 0$ .

In addition to (6) consider the quadratic functional defined by

$$m_a[u] = \int_{R_a} \left[ \sum_{i,j} a_{ij} D_i u D_j u - 2u \sum_i b_i D_i u - cu^2 \right] dx,$$

whose Euler-Jacobi operator is  $l$  and let  $m[u] = \lim_{a \rightarrow \infty} m_a[u]$  (whenever the limit exists). The domain  $\mathfrak{D}_m$  of  $m$  consists of all real-valued continuous functions  $u$  in  $R \cup P$  such that  $u$  has uniformly continuous first partial derivatives in  $R_a$  for all  $a \geq a_0$ ,  $m[u]$  exists, and  $u$  vanishes on  $P$ . The variation of  $m[u]$  is defined as  $V[u] = m[u] - M[u]$ , that is

$$(12) \quad V[u] = \int_R \left[ \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u - 2u \sum_i (b_i - B_i) D_i u + (C - c - G) u^2 \right] dx,$$

with domain  $\mathfrak{D} = \mathfrak{D}_m \cap \mathfrak{D}_M$ . The analogues of (8), (9) for the operator  $l$  are

$$\begin{aligned} \{u, v\}_a &= \int_{C_a} u \sum_{i,j} a_{ij} n_i D_j v \, ds, \\ \{u, v\} &= \lim_{a \rightarrow \infty} \{u, v\}_a. \end{aligned}$$

**THEOREM 1.** *Suppose  $G$  satisfies (5). If there exists a nontrivial solution  $u \in \mathfrak{D}$  of  $lu = 0$  such that  $\{u, u\} \leq 0$  and  $V[u] > 0$ , then every solution  $v$  of  $Lv = 0$  for which*

$[u^2/v, v] \geq 0$  vanishes at some point of  $R \cup P$ . The same conclusion holds if the conditions  $V[u] > 0$ ,  $[u^2/v, v] \geq 0$  are replaced by  $V[u] \geq 0$ ,  $[u^2/v, v] > 0$  respectively.

**Proof.** Since  $u=0$  on  $P_a$ , it follows from Green's formula that

$$m_a[u] = - \int_{R_a} ulu \, dx + \{u, u\}_a.$$

Since  $lu=0$  and  $\{u, u\} \leq 0$ , we obtain in the limit  $a \rightarrow \infty$  that  $m[u] \leq 0$ . The hypothesis  $V[u] > 0$  is equivalent to  $M[u] < m[u]$ . Hence the condition  $M[u] < 0$  of Lemma 1 is fulfilled and  $v$  vanishes at some point of  $R \cup P$ . The second statement of Theorem 1 follows upon obvious modification of the inequalities.

**THEOREM 2 (SELF-ADJOINT CASE).** Suppose  $b_i = B_i = 0$  in (1) and (2),  $i = 1, 2, \dots, n$ . If there exists a nontrivial solution  $u \in \mathfrak{D}$  of  $lu=0$  such that  $\{u, u\} \leq 0$  and  $V[u] \geq 0$ , then every solution  $v$  of  $Lv=0$  for which  $[u^2/v, v] \geq 0$  vanishes at some point of  $R \cup P$ .

This follows from Lemma 2 by a proof analogous to that of Theorem 1.

In the case that equality holds in (5), that is

$$G = - \sum_{i=1}^n B_i B_i^* / \det(A_{ij}),$$

we define

$$\delta = \sum_{i=1}^n D_i(b_i - B_i) + C - c - G.$$

It follows from (12) by partial integration that

$$(13) \quad V[u] = \int_R \left[ \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u + \delta u^2 \right] dx + \Omega,$$

where

$$\Omega = \lim_{a \rightarrow \infty} \int_{C_a} \sum_i (B_i - b_i) u^2 n_i \, ds.$$

$L$  is called a "strict Sturmian majorant" of  $l$  when the following conditions hold: (i)  $(a_{ij} - A_{ij})$  is positive semidefinite and  $\delta \geq 0$  in  $R$ ; (ii)  $\Omega \geq 0$ ; and (iii) either  $\delta > 0$  at some point in  $R$  or  $(a_{ij} - A_{ij})$  is positive definite and  $c \neq 0$  at some point. A function defined in  $R$  is said to be of class  $C^{2,1}(R)$  when all of its second partial derivatives exist and are Lipschitzian in  $R$ .

**THEOREM 3.** Suppose  $L$  is a strict Sturmian majorant of  $l$  and all the coefficients  $a_{ij}$  involved in  $l$  are of class  $C^{2,1}(R)$ . If there exists a nontrivial solution  $u \in \mathfrak{D}$  of  $lu=0$  such that  $\{u, u\} \leq 0$ , then every solution  $v$  of  $Lv=0$  for which  $[u^2/v, v] \geq 0$  vanishes at some point of  $R \cup P$ .

**Proof.**  $V[u]$  exists since  $u \in \mathfrak{D}$ , and hence each term on the right side of (13)

exists by the strict Sturmian hypothesis. Since  $a_{ij} \in C^{2,1}(R)$ ,  $i, j = 1, 2, \dots, n$ , Aronszajn's unique continuation theorem [1] guarantees that the nontrivial solution  $u$  cannot vanish identically in any open subset of  $R$ . In the case that  $\delta > 0$  at some point in  $R$  it then follows from (13) that  $V[u] > 0$ . In the case that  $\delta \equiv 0$  in  $R$  it follows from (13) and the positive definite hypothesis on  $(a_{ij} - A_{ij})$  that  $V[u] = 0$  only if  $D_i u = 0$  for each  $i = 1, 2, \dots, n$  in some open set  $S$  of  $R$ , that is,  $u$  is constant in  $S$ . Since  $c \neq 0$  at some point  $x_0 \in S$ , the differential equation (1) would not be satisfied at  $x_0$ . Hence  $V[u] > 0$  also in the case that  $\delta \equiv 0$ . The conclusion of Theorem 3 then follows from Theorem 1.

**THEOREM 4 (SELF-ADJOINT CASE).** *Suppose  $b_i = B_i = 0$  in (1) and (2),  $i = 1, 2, \dots, n$ ,  $C \geq c$ , and  $(a_{ij} - A_{ij})$  is positive semidefinite in  $R \cup P$ . If there exists a nontrivial solution  $u \in \mathfrak{D}$  of (1) such that  $\{u, u\} \leq 0$ , then every solution  $v$  of (2) for which  $[u^2/v, v] \geq 0$  vanishes at some point of  $R \cup P$ .*

This is an immediate consequence of Theorem 2. We assert that the same conclusion holds even if  $(A_{ij})$  is only positive *semidefinite*, provided  $L$  is a strict Sturmian majorant of  $l$  and all the coefficients  $a_{ij}$  are of class  $C^{2,1}(R)$ . In fact, under these assumptions  $V[u] > 0$  as in Theorem 3, i.e.,  $M[u] < 0$  by the proof of Theorem 1, and Lemma 2 remains valid for positive semidefinite  $(A_{ij})$  provided the hypothesis  $M[u] \leq 0$  is replaced by  $M[u] < 0$ .

With trivial modifications the above theorems and proofs remain valid in the case that  $R$  is a *bounded* domain, i.e.,  $C_a$  is void for  $a \geq a_0$ . In particular Theorem 2 implies Theorem A and Theorem 1 implies the author's recent result [7] for the general elliptic equations (1), (2) on bounded domains.

In the case  $n=2$  considered by Protter [6], the condition  $\delta \geq 0$  of Theorem 3 reduces to

$$(A_{11}A_{22} - A_{12}^2) \left( \sum_{i=1}^2 D_i(b_i - B_i) + C - c \right) \geq A_{11}B_2^2 - 2A_{12}B_1B_2 + A_{22}B_1^2.$$

If  $R$  is a bounded domain, Theorem 3 then reduces (with trivial modifications) to the author's result in [7].

It is interesting to note the following one-dimensional instances of Theorem 2, in which  $R$  is an open interval  $(\alpha, \beta)$ . When  $n=1$  and  $b_1 = B_1 = 0$ , the differential equations (1), (2) have the form

$$(14) \quad (au')' + cu = 0, \quad a > 0,$$

$$(15) \quad (Av')' + Cv = 0, \quad A > 0.$$

**THEOREM 5.** *If there exists a nontrivial solution  $u$  of (14) in  $(\alpha, \infty)$  such that  $u(\alpha) = 0$ ,  $a(x)u(x)u'(x) \rightarrow 0$  as  $x \rightarrow \infty$ , and*

$$(16) \quad \int_{\alpha}^{\infty} [(a-A)u'^2 + (C-c)u^2] dx \geq 0,$$

then every solution  $v$  of (15) for which  $A(x)u^2(x)v'(x)/v(x)$  has a nonnegative limit as  $x \rightarrow \infty$  has a zero on  $[\alpha, \infty)$ . Unless  $v$  is a constant multiple of  $u$ ,  $v$  has a zero in  $(\alpha, \infty)$ .

**Proof.** The first statement follows immediately from Theorem 2. To prove the second statement, recall from the proof of Lemma 1 that for all  $a \geq a_0$ ,

$$(17) \quad \int_{\alpha}^a F[u] dx = \left[ \frac{A(x)u^2(x)v'(x)}{v(x)} \right]_{\alpha}^a + \int_{\alpha}^a Av^2 \left( \frac{u}{v} \right)'^2 dx.$$

Since the solutions of second order ordinary linear differential equations have only simple zeros, an application of L'Hospital's rule yields

$$\lim_{x \rightarrow \alpha} \frac{A(x)u^2(x)v'(x)}{v(x)} = 0.$$

Thus the limit of the first term on the right side of (17) as  $a \rightarrow \infty$  is nonnegative. The second term is nonnegative for all  $a$  and zero iff  $u$  is a constant multiple of  $v$ . Hence  $M[u] > 0$  unless  $v$  is a constant multiple of  $u$ . This contradicts the hypothesis (16).

The next result applies to the case that  $\alpha, \beta$  may be singular points of the differential equations (14), (15); the possibility that they are  $\pm \infty$  is not excluded. The proof is similar to that of Theorem 5 and will be omitted.

**THEOREM 6.** *If there exists a nontrivial solution  $u$  of (14) in  $(\alpha, \beta)$  such that  $a(x)u(x)u'(x) \rightarrow 0$  as  $x \rightarrow \alpha$  and as  $x \rightarrow \beta$ , and*

$$(18) \quad \int_{\alpha}^{\beta} [(a-A)u'^2 + (C-c)u^2] dx > 0,$$

then every solution  $v$  of (15) for which  $A(x)u^2(x)v'(x)/v(x)$  has a nonnegative limit as  $x \rightarrow \beta$  and a nonpositive limit as  $x \rightarrow \alpha$  has a zero in  $(\alpha, \beta)$ . If the left side of (18) is only nonnegative, the same conclusion holds unless  $v$  is a constant multiple of  $u$ .

In the special case that  $\alpha, \beta$  are ordinary points of (14) and (15), this reduces to the following generalization of the classical Sturm-Picone theorem; our result is a slight extension of Leighton's theorem [5].

**THEOREM 7.** *If there exists a nontrivial solution  $u$  of (14) in  $[\alpha, \beta]$  such that  $u(\alpha) = u(\beta) = 0$  and the left side of (18) is nonnegative, then every solution of (15) except a constant multiple of  $u$  has a zero in  $(\alpha, \beta)$ .*

As an example of Theorem 5, consider the differential equations

$$(19) \quad u'' + (2n+1-x^2)u = 0,$$

$$(20) \quad v'' + [2n+1-x^2+p(x)]v = 0,$$

on a half-open interval  $[\alpha, \infty)$ , where  $p(x)$  is a polynomial. Equation (19) has the well-known solution  $u(x) = \exp(-x^2/2)H_n(x)$ , where  $H_n(x)$  denotes the Hermite

polynomial of degree  $n$ . Clearly  $u \in \mathfrak{D}$ . Since every solution  $v$  of (20) satisfies  $v'(a)/v(a) \sim q(a)$  as  $a \rightarrow \infty$ , where  $q(a)$  is a polynomial, it follows that the hypothesis  $u^2(a)v'(a)/v(a) \rightarrow 0$  as  $a \rightarrow \infty$  is fulfilled. Hence if  $\alpha$  is a zero of  $H_n(x)$ , then every solution of (20) has a zero in  $(\alpha, \infty)$  provided

$$\int_{\alpha}^{\infty} p(x)u^2(x) dx > 0.$$

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